Appendix 1

Tail dependence for copulas

Tail dependence is a measure of concordance between less probable values of variables. This concordance tends to concentrate on the lower and upper tails of the joint distribution.

In a bivariate context, let F_i be the marginal distribution function of a random variable X_i $(i=1,2)$ and let u be a threshold value; then the lower tail dependence coefficient, λ_L , is defined as

$$\lambda_L = \lim_{u\to 0^+} P\left\{X_2 \le F_2^{-1}(u) \mid X_1 \le F_1^{-1}(u)\right\}$$

and, hence

$$P\left\{X_2 \le F_2^{-1}(u) \middle| X_1 \le F_1^{-1}(u)\right\} = \frac{P\left\{X_1 \le F_1^{-1}(u), X_2 \le F_2^{-1}(u)\right\}}{P\left\{X_1 \le F_1^{-1}(u)\right\}} = \frac{C(u,u)}{u}.$$

Then, an alternative definition, in terms of copula function, is

$$\lambda_L = \lim_{u\to 0^+}\left\{\frac{C(u,u)}{u}\right\}.$$

In a similar way, the upper tail dependence is given by,

$$\lambda_U = \lim_{u\to 1^-} P\left\{X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)\right\}$$

For $\lambda_U \in (0,1]$, X_1 and X_2 are asymptotically dependent on the upper tail; if λ_U is null, X_1 and X_2 are asymptotically independent.

Hence,

$$P\left\{X > F_2^{-1}(u) \middle| X_1 > F_1^{-1}(u)\right\} = \frac{1 - P\left\{X_1 \le F_1^{-1}(u)\right\} - P\left\{X_2 \le F_2^{-1}(u)\right\} + P\left\{X_1 \le F_1^{-1}(u), X_2 \le F_2^{-1}(u)\right\}}{1 - P\left\{X_1 \le F_1^{-1}(u)\right\}}.$$

Then, it is possible to recur to an alternative and equivalent definition, for continuous random variables, from which it is clear that the concept of tail dependence is indeed a copula property (Joe, 1997)

$$\lambda_U = \lim_{u\to 1^-}\frac{\hat{C}(1-u,1-u)}{1-u} = \lim_{u\to 1^-}\left\{\frac{1-2u+C(u,u)}{1-u}\right\}.$$

Where $\hat{C}$ is the survival copula function defined as

$$\hat{C}(1-u_1,1-u_2) = 1 - P\left\{X_1 \le F_1^{-1}(u)\right\} - P\left\{X_2 \le F_2^{-1}(u)\right\} + P\left\{X_1 \le F_1^{-1}(u), X_2 \le F_2^{-1}(u)\right\} =$$
$$= 1 - P(U_1 \le u_1) - P(U_2 \le u_2) + P(U_1 \le u_1, U_2 \le u_2).$$

It is simple to show that $\hat{C}$ is strictly related to the copula function through the following relationship

$$\hat{C}(1-u_1, 1-u_2) = 1-u_1-u_2+C(u_1,u_2).$$

A multivariate generalization of the tail dependence coefficients (De Luca and Rivieccio, 2009) consists in to consider *h* variables and the conditional probability associated to the remaining *n - h* var iables, given, respectively, by

$$\lambda_L^{1\ldots h|h+1\ldots n} = \lim_{u\to 0^+} P(F_1(X_1)\le u,\ldots,F_h(X_h)\le u \,|\, F_{h+1}(X_{h+1})\le u,\ldots,F_n(X_n)\le u)$$
$$= \lim_{u\to 0^+}\left\{\frac{C_n(u,\ldots,u)}{C_{n-h}(u,\ldots,u)}\right\}.$$

Indeed, the upper (lower) tail dependence coefficient can be interpreted as the probability of very high (low) returns for *h* assets provided that very high (low) returns have occurred for the remaining $n-h$ assets.

$$\lambda_U^{1\ldots h|h+1\ldots n} = \lim_{u\to 1^-} P(F_1(X_1)> u,\ldots,F_h(X_h)> u \,|\, F_{h+1}(X_{h+1})> u,\ldots,F_n(X_n)> u)$$
$$= \lim_{u\to 1^-}\left\{\frac{\hat{C}_n(1-u,\ldots,1-u)}{\hat{C}_{n-h}(1-u,\ldots,1-u)}\right\}.$$

Non parametric tail dependence measures

In order to select an adequate copula function able to capture accurately the dependence structure showed by co-movements of extreme return pair-wise, can be useful to estimate the empirical tail dependence by mean of non-parametric method.

The non-parametric bivariate coefficient of lower tail dependence, λ_L^{NP}, can be obtained as (De Luca and Rivieccio, 2009)

$$\lambda_L^{NP}(k) = P(X_2 \le x_2^* | X_1 \le x_1^*),$$

or conversely, where x_i^* is assumed to be $\mu_i - k\sigma_i$. This statistic depends on *k.*

The concept of bivariate upper tail dependence is defined in a similar way as

$$\lambda_U^{NP}(k) = P(X_2 > x_2^* | X_1 > x_1^*),$$

where x_i^* is assumed to be $\mu_i + k\sigma_i$.